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THE UNSTEADY TEMPERATURE FIELD IN PLANE COUETTE FLOW

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Abstract—The viscosity of a fluid in a Couette flow system can not be measured isothermally because dissipation generates heat which has to be removed by conduction to the walls. The present paper investigates the development of the temperature field from its isothermal initial state to the final equilibrium condition. This knowledge is necessary for a judgment whether the viscosity can be measured at a time at which the viscosity field is practically developed whereas the heat generated by dissipation is still negligible. The developing temperature and velocity fields are calculated for a Newtonian fluid with a temperature dependent viscosity assuming a locally and timewise constant wall temperature. A parameter (βBr) characterizing the development of the temperature field determines how rapidly the temperature field develops and whether an equilibrium condition exists at all.

NOMENCLATURE

a,	thermal diffusivity [cm²/s];
$b_n(Fo)$, $T_n(Fo)$,	Fourier-coefficients, see equations (23) and (29);
c,	heat capacity [kcal/kggrd];
h,	slot width [cm];
k,	dimensionless coordinate, y/h;
η,	viscosity [kp . s/cm²];
T,	temperature [grd];
t,	time [s];
U,	unsteady part of temperature field, $\theta - \theta_\infty$;
v, V,	velocity in x direction [cm/s];
x, y,	coordinates [cm];
β,	temperature coefficient ($\beta > 0$ for liquids);
λ,	thermal conductivity [kcal/cmsgrd];
θ	dimensionless temperature, $1 + \beta(T - T_0)/T_0$;
ρ,	density [kg/cm³];
τ,	shear in x direction on a plane y = constant [kp/cm²].

Indices

$^\circ$,	initial state;
$^\infty$,	equilibrium state;
w,	at wall surface;
Br,	Brinkmann number, $\tau_\infty^2 h^2/\eta_0 \lambda T_0$;
Fo,	Fourier number, ta/h^2;
Pr,	Prandtl number, $\eta/a\rho$.

INTRODUCTION

THE MEASUREMENT of the viscosity of a fluid by a device modeling Couette flow is made difficult by the fact that it is impossible to maintain isothermal conditions. The dissipation in the fluid develops heat which has to be conducted to the walls. As a consequence a temperature field exists in the fluid at steady operation.

This temperature field does not develop instantaneously. Its establishment requires a certain time the magnitude of which depends on the fluid involved and on the experimental conditions. Bartenew and Kusnetschikowa [1] measured the required starting time for various

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kinds of Butadienkautschuk and found values of order of magnitude of minutes. (The width of the fluid layer was 1 and 3 mm and the shear velocity between 0·01 and 26·5 s^{-1}.)

A measurement of the viscosity under isothermal conditions must, therefore, be performed at a time after the starting of the device which is long enough for the velocity field to develop fully but which is sufficiently short so that the heat developed by dissipation is still negligible. It will be demonstrated that for a Newtonian fluid the Prandtl number is an important criterion establishing the condition under which the kinematic starting process is finished before the thermal development begins.

Krekel (8) measured the shear stress and the temperature of the fluid during the starting of a Couette system, using silicone oil as a model fluid. For operation with constant velocity he found that with increasing values of $\beta \cdot \mathrm{Br}$ the fluid temperature increased more rapidly, which in turn caused a more rapid and greater decrease of the shear stress. Krekel calculated the temperature and the velocity field for the operation with constant shear stress and used this solution as an approximation for the operation with constant velocity. The shear dependence of the viscosity was taken into account by the sinh-formula and the temperature dependence was taken to be linear.

Powell and Middleman [2] investigated the thermal development for a Newtonian fluid with constant viscosity with the goal to establish the time during which the uncooled walls of the Couette system remain practically isothermal and the time required for the development of a constant temperature field in the walls. The restriction to a temperature independent viscosity had the consequence that the times obtained were a function of geometry and thermal conductivity only and that the Brinkmann number had no influence.

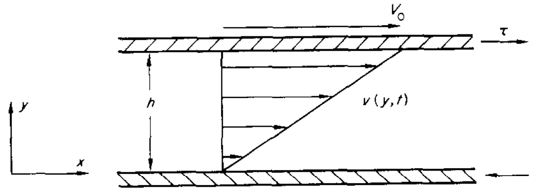

FIG. 1. Plane Couette flow.

In this paper, the development of the temperature field is calculated for plane shear flow of a Newtonian fluid of variable viscosity with the assumption that the walls remain at constant temperature.

System of equations

The fluid considered in this paper is assumed to be incompressible and Newtonian and to have constant thermal conductivity and diffusivity. The temperature dependence of the viscosity is approximated by a linear function. The solution of the velocity field has therefore to be restricted to temperature ranges for which this approximation applies.

The Couette flow is generated between two plane walls (Fig. 1). Dynamic equilibrium requires

$$\rho \frac{\partial v(y,t)}{\partial t} = \frac{\partial \tau(y,t)}{\partial y}, \qquad (1)$$

where the velocity has only a component in x direction. τ denotes the shear stress acting in x direction on planes $y = \text{const}$.

The relationship between shear stress and viscosity is

$$\tau(y,t) = \eta(T) \frac{\partial v(y,t)}{\partial y}. \qquad (2)$$

The viscosity in this equation depends indirectly on the y coordinate.

The conservation of energy requires

$$\rho c_v \frac{\partial T(y,t)}{\partial t} = \lambda \frac{\partial^2 T(y,t)}{\partial y^2} + \tau(y,t) \frac{\partial v(y,t)}{\partial y}. \qquad (3)$$

The boundary and initial conditions for the temperature field are

isothermal wall:

$$T(y=0,t) = T(y=h,t) = T_W \qquad (4)$$

uniform initial temperature:

$$T(y, t=0) = T_0 = T_W.$$

The temperature dependence of the viscosity will be expressed by the equation

$$\frac{1}{\eta} = \frac{1}{\eta_0}\left(1 + \beta\frac{(T - T_0)}{T_0}\right) \quad (5)$$

in which η_0 denotes the viscosity at the initial temperature T_0. The equations (2), (3) and (5) can be combined to the expression

$$\frac{1}{a}\frac{\partial\theta(y,t)}{\partial t} = \frac{\partial^2\theta(y,t)}{\partial y^2} + \frac{\tau^2(y,t).h^2}{\lambda\eta_0}\theta(y,t)$$

with

$$\theta = 1 + \beta\frac{(T - T_0)}{T_0}. \quad (6)$$

Non-dimensionalization

The dimensionless parameter characterizing the dynamic process, equation (1), is the product of Fourier and Prandtl numbers

$$Fo\,.\,Pr = \frac{t\eta_0}{\rho h^2}. \quad (7)$$

The development of the temperature field, equation (3), is characterized by the dimensionless parameter

$$Fo = \frac{ta}{h^2} \quad (8)$$

$$Br = \frac{\tau_\infty^2 h^2}{\eta_0\lambda T_0}. \quad (9)$$

Fo and $Fo\,.\,Pr$ are actually dimensionless times. The Brinkmann number Br is the ratio of the heat generated by dissipation to the heat conducted away. In addition a dimensionless coordinate

$$k = y/h \quad (10)$$

will be introduced. With these notations the temperature field can be expressed in the following form

$$\frac{\partial\theta(k,Fo)}{\partial Fo} = \frac{\partial^2\theta(k,Fo)}{\partial k^2} + \frac{\tau^2(k,Fo)}{\tau_\infty^2}\beta Br\theta\,(k,Fo),$$

$$0 \leqslant k \leqslant 1,\ 0 < Fo, \quad (11)$$

b.c.: $\theta(0,Fo) = \theta(1,Fo) = 1,\ 0 < Fo$

i.c.: $\theta(k,0) \ = 1,\ 0 \leqslant k \leqslant 1.$

Equation (11) can be solved exactly when the ratio is constant. This holds for a starting process with constant shear. For a starting process with constant relative velocity

$$v(1,Fo) = V_0$$

the magnitude of $\tau(k,Fo)/\tau_\infty$ will be estimated in the following paragraphs.

Kinematic starting without heat dissipation

The calculation of the temperature field will be based on the assumption that the kinematic starting process is practically finished before the heat dissipation has generated an amount of heat which is not negligible. The shear stress in the field is then uniform

$$\tau(k,Fo) \rightarrow \tau(Fo).$$

The kinematic starting process will be calculated at first in order to justify this assumption:

At the beginning of the experiment, the plates

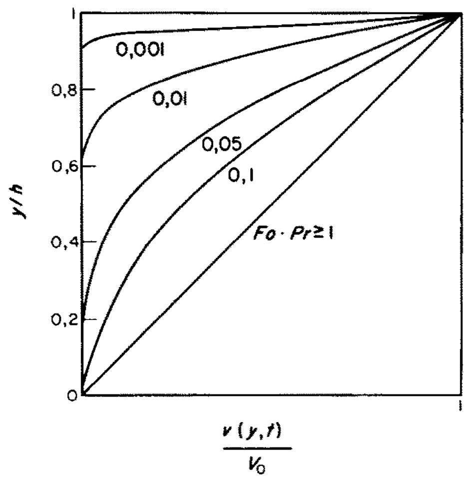

FIG. 2. Velocity field during kinematic starting with constant velocity. The parameter $Fo\,.\,Pr$ is a dimensionless time.

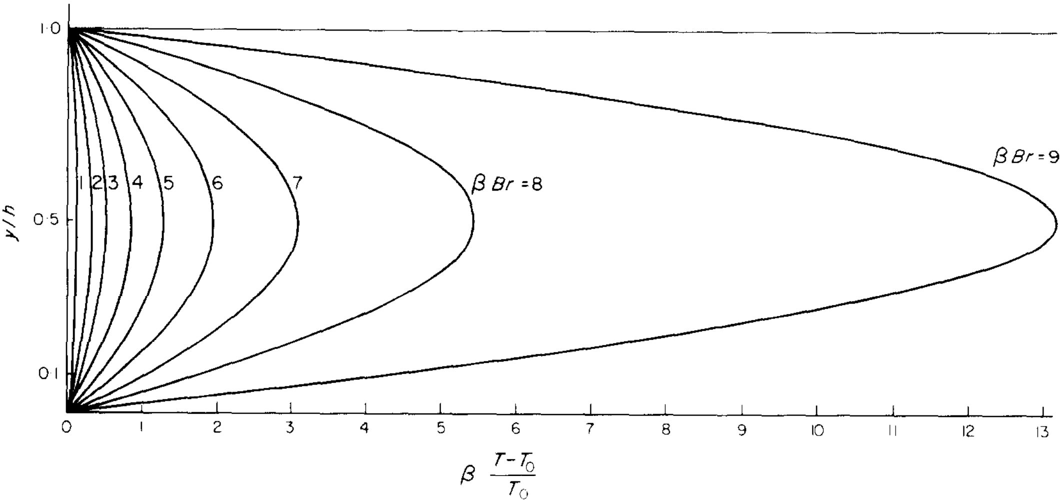

FIG. 3. Equilibrium temperature field with $\beta \, . \, Br$ as parameter.

and the fluid are at rest. At the time $t = 0$ (or $Fo = 0$) the upper plate starts moving with a constant velocity V_0. In the course of time the momentum "diffuses" from the moving plate towards the plate at rest. The liquid layers in this way accelerate towards the equilibrium velocity (Fig. 2). The velocity field is obtained as a solution of the Navier–Stokes equation, equation (1) with (2). (For the calculation see [3]).

$$\frac{v(k, Fo)}{V_0} = k + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n}$$

$$\exp(-Fo \, . \, Prn^2\pi^2) \sin(n\pi k). \qquad (12)$$

At a time which corresponds to $Fo \, . \, Pr = 1$, the shear stress has approached the equilibrium shear stress to 99·99 per cent. This means for the assumption above that the thermal starting process at

$$Fo = \frac{1}{Pr}$$

should not yet have started. It also indicates that the calculation in the next paragraph applies to

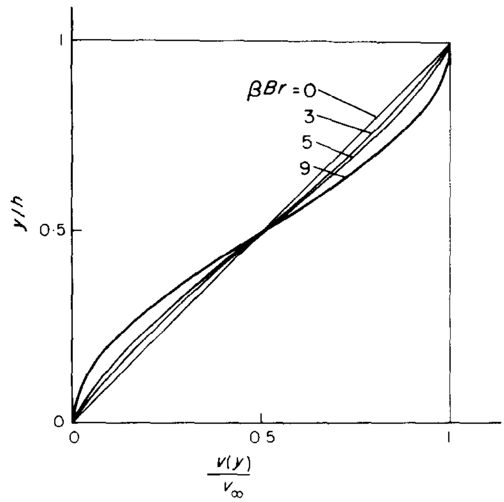

FIG. 4. Equilibrium velocity field with $\beta \, . \, Br$ as parameter. (a) Velocity referred to V_∞.

liquids with large Prandtl numbers only. From Fig. 9, which will be discussed later on, one can determine when this condition applies.

An analogous computation for the kinematic starting for constant shear stress condition on the moving plate leads to the same criterion. The velocity field for this case is shown by McKelvey [4].

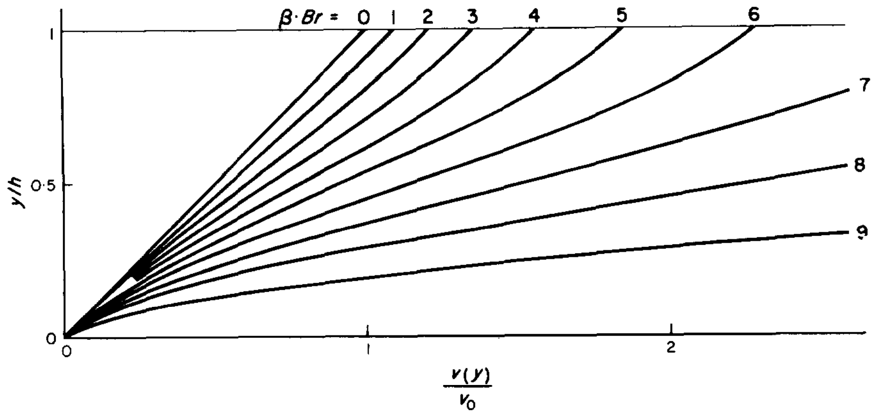

FIG. 4. (b) Velocity referred to V_0.

Starting condition for the thermal development

The kinematic starting process is considered as finished. This means that an established velocity field is assumed

$$\frac{v(k,0)}{V_0} = k \tag{13}$$

and that the shear stress is

$$\tau_0 = \eta_0 \frac{V_0}{h}. \tag{14}$$

The viscosity is constant since the field is still isothermal.

$$T - T_w = 0. \tag{15}$$

Equilibrium temperature field and the corresponding stress and velocity field

The heat generated by dissipation causes the fluid temperature to increase. Heat is removed by conduction to the walls which are kept at the original temperature. Under steady condition the temperature field is such that all of the heat generated by dissipation is removed by conduction. This equilibrium temperature field has been calculated by various authors e.g. [5–10]. For plane Couette flow, the temperature field is obtained from the energy equation, equation (11), with the conditions

$$\frac{\partial\theta}{\partial Fo} = 0, \qquad \tau = \tau_\infty = \text{const.}$$

$$\theta_\infty(k) = \frac{1-\cos\sqrt{\beta Br}}{\sin\sqrt{\beta Br}}\sin\sqrt{\beta Br}k + \cos\sqrt{\beta Br}k.$$

The temperature and velocity fields for steady condition are presented in Figs. 3 and 4.

The ratio of the relative velocity for equilibrium condition to that at initiation of the movement is for the starting process with constant shear stress given by the following equation

$$\frac{v(1,Fo\to\infty)}{v(1,0)} = \frac{V_\infty}{V_0} = \frac{2-2\cos\sqrt{\beta Br}}{\sqrt{\beta Br}\sin\sqrt{\beta Br}}. \tag{17}$$

For the start with constant relative velocity, the ratio of the shear stresses is

$$\frac{\tau(Fo\to\infty)}{\tau(0)} = \frac{\tau_\infty}{\tau_0} = \frac{\sqrt{\beta Br}\sin\sqrt{\beta Br}}{2-2\cos\sqrt{\beta Br}}. \tag{18}$$

The solution for the temperature field can be composed from the equilibrium solution and from a transitional contribution

$$\theta(k,Fo) = \theta_\infty(k) + U(k,Fo). \tag{19}$$

Introducing this equation into the original differential equation (11) results in the following equation for the transitional contribution

$$\frac{\partial U(k, Fo)}{\partial Fo} = \frac{\partial^2 U(k, Fo)}{\partial k^2} + \frac{\tau^2(Fo)\,\beta Br}{\tau_\infty^2}$$

$$U(k, Fo) + \left(\frac{\tau^2(Fo)}{\tau_\infty^2} - 1\right)\beta Br\,\theta_\infty(k),$$

$$0 \leqslant k \leqslant 1,\ 0 < Fo, \qquad (20)$$

b.c.: $U(0, Fo) = U(1, Fo) = 0, \quad 0 < Fo,$
i.c.: $U(k, 0) = 1 - \theta_\infty(k), \quad 0 \leqslant k \leqslant 1.$

Temperature field for operation with constant shear stress

The experiments with constant shear stress require

$$\tau = \tau_0 = \tau_\infty = \text{const.}$$

Equation (20) is being reduced to a linear differential equation with constant coefficients which is solvable by a Fourier analysis.

The temperature field is

$$\frac{T(k, Fo) - T_0}{T_0} = \frac{1}{\beta}\left[\frac{1 - \cos\sqrt{\beta Br}}{\sin \beta Br}\sin\sqrt{\beta Br}k + \cos\sqrt{\beta Br}k - 1 + \sum_{n=1}^{\infty} b_n(Fo)\cdot\frac{\sin(n\pi k)}{n\pi}\right]. \qquad (21)$$

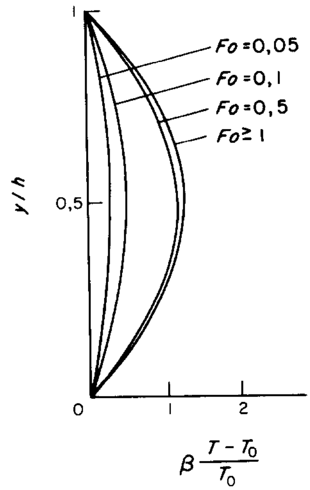

FIG. 5. Temperature field during thermal starting with constant shear (for $\beta \cdot Br = 5$).

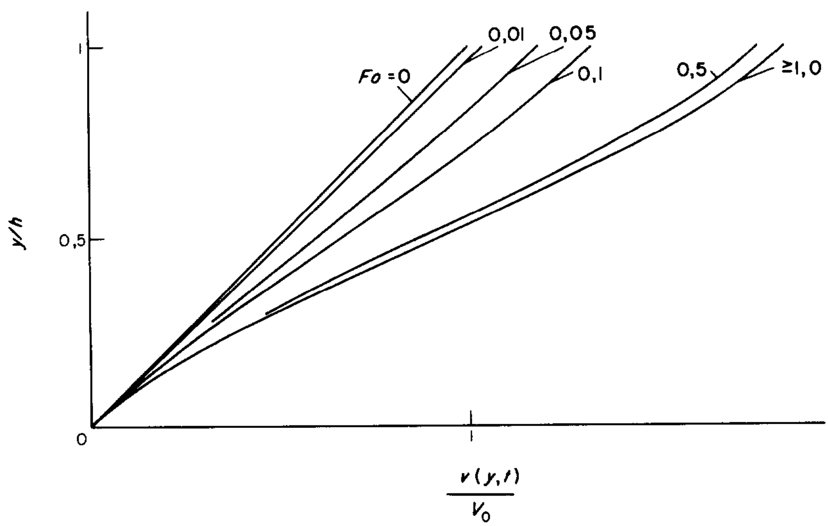

FIG. 6. Velocity field during thermal starting with constant shear (for $\beta \cdot Br = 5$). The velocity is normalized with the starting velocity V_0.

From the temperature field the velocity field is calculated by integration

$$\frac{v(k, Fo)}{V_0} = \frac{\tau(Fo)}{\tau_0} \int_0^k \theta(Fo, k)\,dk$$

$$= \frac{(1 - \cos\sqrt{\beta Br})(1 - \cos\sqrt{\beta Br}k)}{\sqrt{\beta Br}\sin\sqrt{\beta Br}} + \frac{\sin\sqrt{\beta Br}k}{\sqrt{\beta Br}}$$

$$+ \sum_{n=1}^{\infty} b_n(Fo)\,.\frac{(1 - \cos n\pi k)}{n^2\pi^2}. \quad (22)$$

The Fourier coefficients b_n are obtained from the starting condition

$$b_n = [1 - (-1)^n]\frac{2\,.\,\beta Br}{\beta Br - n^2\pi^2} \times \exp[Fo(\beta Br - n^2\pi^2)]. \quad (23)$$

The temperature field (multiplied by the temperature coefficient β) is presented in Fig. 5 for the parameter value $\beta Br = 5$ and for various Fourier numbers. The velocity fields normalized by the starting velocity V_0 and by the equilibrium velocity V_∞ respectively are presented in Figs. 6 and 7.

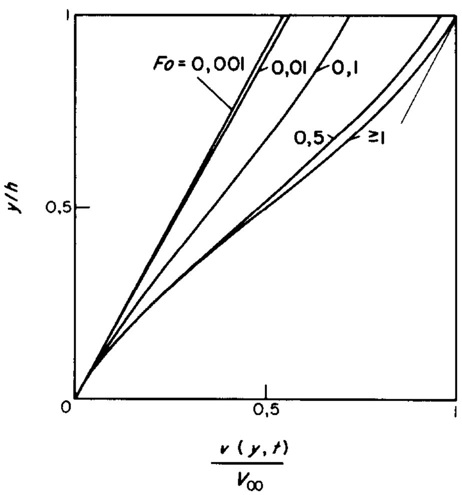

FIG. 7. Velocity field during thermal starting with constant shear (for $\beta\,.\,Br = 5$). The velocity is normalized with the equilibrium velocity V_∞.

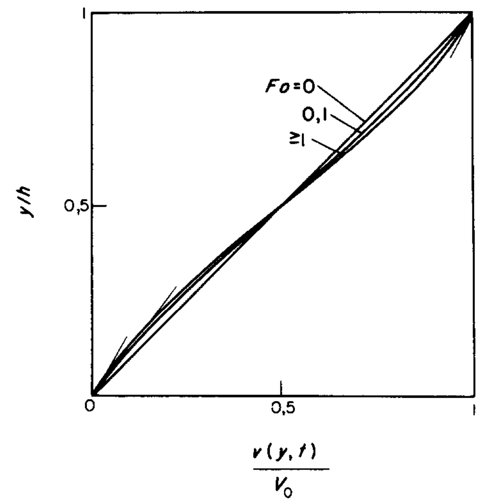

FIG. 8. Velocity field during thermal starting with constant relative velocity V_0.

Temperature field for operation with constant velocity

The initial and the final state of the temperature field is already known, equations (15) and (16). The temperature field in the neighbourhood of the equilibrium state is approximated by equation (17). Integration results in the velocity field (Fig. 8)

$$\frac{v(k, Fo)}{V_0} = \frac{\tau(Fo)}{\tau_0} \times \left[\frac{(1 - \cos\sqrt{\beta Br})(1 - \cos\sqrt{\beta Br}k)}{\sqrt{\beta Br}\sin\sqrt{\beta Br}} + \frac{\sin\sqrt{\beta Br}}{\sqrt{\beta Br}} + \sum_{n=1}^{\infty} b_n(Fo)\frac{1 - \cos(n\pi k)}{n^2\pi^2}\right]. \quad (24)$$

The local shear stress is described by the equation

$$\frac{\tau(Fo)}{\tau_0} = \left[\frac{2 - 2\cos\sqrt{\beta Br}}{\sqrt{\beta Br}\sin\sqrt{\beta Br}} \sum_{n=1}^{\infty} b_n(Fo)\,.\frac{1 - (-1)^n}{n^2\pi^2}\right]^{-1}. \quad (25)$$

We are, however, especially interested in the deviation from the isothermal state. Therefore, an approximate solution must be found for small Fourier numbers. For this purpose, the shear stress is assumed constant in consecutive intervals. A first approximation is obtained from equations (18) and (25). For the following

$$\frac{\tau(Fo)}{\tau_0} = \left[\frac{2 - 2\cos\sqrt{\beta Br}}{\sqrt{\beta Br}\sin\sqrt{\beta Br}} + \sum_n^{\infty} T_n(Fo)\left(\frac{1-(-1)^n}{n^2\pi^2}\right)\right]^{-1}. \quad (28)$$

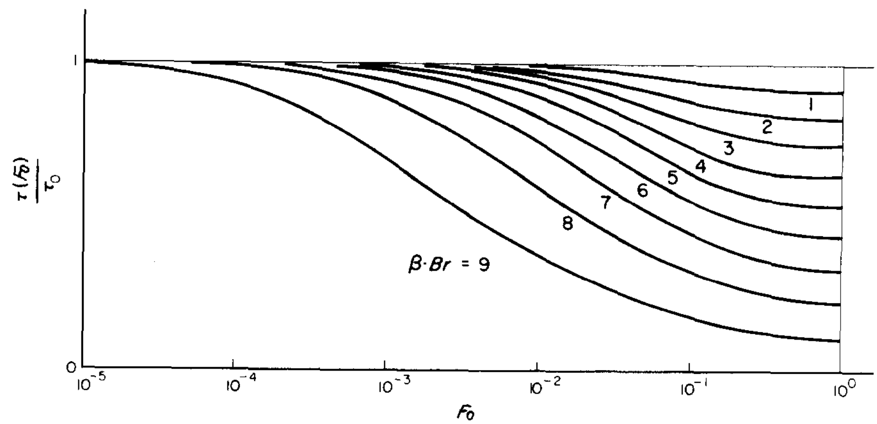

FIG. 9. Shear stress during thermal starting with constant relative velocity V_0. The parameter is $\beta \cdot Br$.

iteration steps the value of the shear stress is taken from equation (28). Approximate solutions for the temperature field, the velocity field, and the shear stress field are

$$\frac{T - T_0}{T_0} = \frac{1}{\beta}\left[\frac{1-\cos\sqrt{\beta Br}}{\sin\sqrt{\beta Br}}\sin\sqrt{\beta Br}k + \cos\sqrt{\beta Br}k - 1 + \sum_{n=1}^{\infty} T_n(Fo)\frac{\sin(n\pi k)}{n\pi}\right], \quad (26)$$

$$\frac{v(k, Fo)}{V_0} = \frac{\tau(Fo)}{\tau_0}\left[\frac{(1-\cos\sqrt{\beta Br})(1-\cos\sqrt{\beta Br}k)}{\sqrt{\beta Br}\sin\sqrt{\beta Br}} + \frac{\sin\sqrt{\beta Br}k}{\sqrt{\beta Br}} + \sum_{n=1}^{\infty} T_n(Fo)\frac{1-\cos(n\pi k)}{n^2\pi^2}\right], \quad (27)$$

The Fourier coefficients are

$$T_n(Fo) = \frac{2\beta Br[1-(-1)^n]}{[\tau(Fo)/\tau_\infty^2]\,\beta Br - n^2\pi^2},$$
$$\left\{\frac{\tau^2(Fo)}{\tau_\infty^2}\exp\left[Fo\left(\frac{\tau^2(Fo)}{\tau_\infty^2}\beta Br - n^2\pi^2\right)\right] + \frac{\{[\tau^2(Fo)/\tau_\infty^2] - 1\}\,n^2\pi^2}{\beta Br - n^2\pi^2}\right\}. \quad (29)$$

The starting process is presented in Fig. 8 for the parameter value $\beta Br = 5$. The deviation from isothermal condition is determined by the deviation of the shear stress ratio from the initial value 1 (Fig. 9).

For large values of the parameter βBr the iteration converges at small values of the Fourier number only. By means of an underrelaxation procedure it was possible to extend the range where the iteration converges up to $\beta Br = 9$.

Range of validity

The solution described by equations (21)–(28) is valid for $0 < \beta . Br < \pi^2$.

At $\beta Br = 0$ the temperature remains at the original level. The time at which the thermal process starts decreases with increasing value of βBr. The solution presented here is therefore applicable for fluids with corresponding high Prandtl numbers only. For $\beta Br \geqslant \pi^2$, the temperature increases continuously with increasing time to higher and higher values. The heat generated by dissipation cannot be conducted to the walls rapidly enough.

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CHAMP INSTATIONNAIRE DE TEMPÉRATURE DANS L'ÉCOULEMENT PLAN DE COUETTE

Résumé—La viscosité d'un fluide dans un système d'écoulement de Couette ne peut être mesurée isothermiquement parce que la dissipation créée de la chaleur qui doit être enlevée par conduction aux parois. Cet article étudie le développement du champ de température à partir de l'état initial jusqu'à l'équilibre final. Cette connaissance est nécessaire pour savoir si la viscosité peut être mesurée à un instant où le champ dynamique est pratiquement développé tandis que la chaleur créée par dissipation est pratiquement négligeable. Les champs thermiques et dynamiques sont calculés pour un fluide newtonien dont la viscosité dépend de la température en supposant une température pariétale constante dans le temps et sur la surface. Un paramètre qui caractérise le développement du champ de temperature fixe la vitesse avec laquelle le champ de température se développement et les conditions d'équilibre.

DAS INSTATIONÄRE TEMPERATURFELD IN EBENER SCHLEPPSTRÖMUNG

Zusammenfassung—Die Viskosität einer Flüssigkeit kann in einem Couettesystem nicht unter isothermen Bedingungen gemessen werden, da durch Dissipation Wärme entsteht, die zu den Wänden hin abgeleitet werden muss. Diese Arbeit untersucht den Aufbau des Temperaturfeldes vom isothermen Ausgangs- zum Gleichgewichtszustand. Die Kenntnis darüber ist zur Beurteilung notwendig, ob die Viskosität zu einem Zeitpunkt gemessen werden kann, zu dem das Geschwindigkeitsfeld zwar ausgebildet, die Wärmeentwicklung durch Dissipation jedoch noch vernachlässigbar ist. Die sich aufbauenden Temperatur- und Geschwindigkeitsfelder wurden für eine Newtonsche Flüssigkeit mit temperaturabhängiger Viskosität für isotherme Wände berechnet. Ein Parameter (β Br), mit dem das sich entwickelnde Temperaturfeld charakterisiert wird, bestimmt, wie schnell sich das Temperaturfeld aufbaut und ob überhaupt eine Gleichgewichtsbedingung existiert.

НЕСТАЦИОНАРНОЕ ТЕМПЕРАТУРНОЕ ПОЛЕ В ПЛОСКОМ ТЕЧЕНИИ КУЭТТА

Аннотация—Вязкость жидкости при куэттовском течении нельзя измерить изотермически, т.к. вследствие диссипации генерируется тепло, которое удаляется за счет теплопроводности через стенки. В данной работе исследуктся развитие температурного поля от изотермического начального до конечного равновесного состояния. Эти сведения необходимы для того, чтобы понять, можно ли измерять вязкости тогда, когда поле вязкости уже практически развито, а тепло, выделяемое за счет диссипации, еще пренебрежимо мало. Развивающиеся поля температуры и скорости рассчитываются для ньютоновской жидкости в предположении локальной зависимости вязкости от температуры и постоянства температуры стенки. Параметр (β *Br*), характеризующий развитие температурного поля, определяет скорость развития температурного поля и существует ли вообще условие равновесия.